Recitation 5. April 6

Focus: linear transformations, change of basis, determinants

A **linear transformation** is a function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have:

$$\boxed{\phi(\boldsymbol{v}+\boldsymbol{w}) = \phi(\boldsymbol{v}) + \phi(\boldsymbol{w})} \qquad \text{and} \qquad \boxed{\phi(\alpha \boldsymbol{v}) = \alpha\phi(\boldsymbol{v})}$$

A linear transformation ϕ can be expressed as a matrix B, with respect to given bases $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ and $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ of $\mathbb{R}^m$: the entry b_{ij} on the i–th row and j–th column of B are such that:

$$\boxed{\phi(x_1\boldsymbol{v}_1 + ... + x_n\boldsymbol{v}_n) = (b_{11}x_1 + ... + b_{1n}x_n)\boldsymbol{w}_1 + ...(b_{m1}x_1 + ... + b_{mn}x_n)\boldsymbol{w}_m}$$

Changing the bases $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ and $\boldsymbol{w}_1, ..., \boldsymbol{w}_m$ will mean different coefficients b_{ij}, and hence a different matrix B, for one and the same function ϕ. The general rule is the **change of basis** formula:

$$\boxed{B = W^{-1}AV}$$

where $V = \left[\ \boldsymbol{v}_1 \mid \ldots \mid \boldsymbol{v}_n\ \right]$, $W = \left[\ \boldsymbol{w}_1 \mid \ldots \mid \boldsymbol{w}_n\ \right]$, and A is the matrix which represents ϕ in the standard basis:

$$\boxed{\phi(\boldsymbol{v}) = A\boldsymbol{v}} \quad \Leftrightarrow \quad \boxed{\phi(x_1\boldsymbol{e}_1 + ... + x_n\boldsymbol{e}_n) = (a_{11}x_1 + ... + a_{1n}x_n)\boldsymbol{e}_1 + ...(a_{m1}x_1 + ... + a_{mn}x_n)\boldsymbol{e}_m}$$

We note that if $\phi(\boldsymbol{v}) = A\boldsymbol{v}$ and $\psi(\boldsymbol{v}) = B\boldsymbol{v}$, then $\phi \circ \psi(\boldsymbol{v}) = (AB)\boldsymbol{v}$. Moreover, $\phi^{-1}(\boldsymbol{v}) = A^{-1}\boldsymbol{v}$, assuming the linear transformation ϕ has an inverse, which is equivalent to A being invertible.

Given a square matrix A, its **determinant** (denoted by $\det A$) is the factor by which the linear transformation $\phi(\boldsymbol{v}) = A\boldsymbol{v}$ scales volumes of regions in $\mathbb{R}^n$. It satisfies the property that:

$$\boxed{\det(AB) = (\det A)(\det B)}$$

A computationally efficient way to compute the determinant is to put A in row echelon form, and set:

$$\boxed{\det A = (-1)^{\#}(\text{product of pivots})}$$

where # is the number of row exchanges that you need to do as you put A in row echelon form. Note the identities:

$$\boxed{\det A^T = \det A} \qquad \boxed{\det A^{-1} = \frac{1}{\det A}} \qquad \boxed{\det(\lambda A) = \lambda^n \det A}$$

for an $n \times n$ matrix A.

1. Recall that the linear transformation "counter-clockwise rotation by an angle α" is represented in the standard basis of $\mathbb{R}^2$ by the matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

If you compose a rotation by angle α with a rotation by angle β, what do you get geometrically? What is the matrix that represents this composition? Can you use this to get formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$?

> **Solution:** The composition should be rotation by an angle $\alpha + \beta$, which is represented by the matrix:
>
> $$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$
>
> On the other hand, the matrix representing the composition should be:
>
> $$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

Setting the two matrices above equal to each other gives us the angle sum formulas:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

2. Determine whether the following maps $\phi_a, \phi_b, \phi_c : \mathbb{R}^3 \to \mathbb{R}^3$ are linear. If so, find a matrix representation of the map in terms of the standard basis of $\mathbb{R}^3$, and then find a matrix representation in terms of the basis:

$$\boldsymbol{v}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \boldsymbol{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) $\phi_a\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ x^2+y^2+z^2 \\ 0 \end{bmatrix}$.

(b) $\phi_b(\boldsymbol{v}) = (\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a}$, where $\boldsymbol{a} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \in \mathbb{R}^3$.

(c) $\phi_c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y-z \\ x+2y \\ y-3z \end{bmatrix}$.

Solution: (a) ϕ_a is not linear, as $\phi_a\left(\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\phi_a\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

(b) ϕ_b is linear. Indeed, for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\psi(\alpha\boldsymbol{v} + \beta\boldsymbol{w}) = \big(\boldsymbol{a}\cdot(\alpha\boldsymbol{v}+\beta\boldsymbol{w})\big)\boldsymbol{a} = \big(\alpha(\boldsymbol{a}\cdot\boldsymbol{v}) + \beta(\boldsymbol{a}\cdot\boldsymbol{w})\big)\boldsymbol{a} = \alpha(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{a} + \beta(\boldsymbol{a}\cdot\boldsymbol{w})\boldsymbol{a} = \alpha\phi(\boldsymbol{v}) + \beta\phi(\boldsymbol{w}).$$

In terms of the standard basis, the matrix representation is

$$X = \boldsymbol{a}\boldsymbol{a}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In terms of the basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, let us form the matrix:

$$V = \left[\ \boldsymbol{v}_1 \mid \boldsymbol{v}_2 \mid \boldsymbol{v}_3\ \right] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

One can compute (e.g. by Gauss-Jordan elimination) that:

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

The change of basis formula tells us that the linear transformation ϕ_b is represented by the following matrix in the new basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$:

$$V^{-1}XV = \frac{1}{4}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

In other words, for any numbers x_1, x_2, x_3, we have:

$$\phi_b(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3) = \frac{(x_2+x_3)(\boldsymbol{v}_2+\boldsymbol{v}_3)}{2} \quad \Leftrightarrow \quad \left(\boldsymbol{a}\cdot\begin{bmatrix} x_1+x_3 \\ x_2-x_1 \\ x_3-x_2 \end{bmatrix}\right)\boldsymbol{a} = \frac{x_2+x_3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The formula on the right is easy to check explicitly.

(c) ϕ_c is linear, as:

$$\phi_c\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so by the linearity of matrix multiplication ϕ_c is linear. By the change of basis formula, the matrix representing ϕ_c in the new basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is:

$$A^{-1}YA = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 1 \\ -1 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 0 & 3 & -1 \end{bmatrix}.$$

In other words, for any numbers x_1, x_2, x_3 we have:

$$\phi_c(x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_3 + x_3\boldsymbol{v}_3) = (2x_1 - 3x_2 + x_3)\boldsymbol{v}_1 + (x_1 - x_2 + 2x_3)\boldsymbol{v}_2 + (3x_2 - x_3)\boldsymbol{v}_3$$

3. Compute the determinant of:

$$M = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \\ 1 & 3 & -1 & 2 \\ -1 & 3 & 0 & 5 \end{bmatrix}$$

by using row operations.

Solution: We first swap the first and third rows, and then the second and fourth rows to arrive at the matrix:

$$M' = \begin{bmatrix} 1 & 3 & -1 & 2 \\ -1 & 3 & 0 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \end{bmatrix}.$$

Therefore $\det M = (-1)^2 \det M' = \det M'$. We now perform elimination operations on M':

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ -1 & 3 & 0 & 5 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \end{bmatrix} \xrightarrow{r_4+2r_3} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

which shows $\det(M') = 1 \cdot 6 \cdot 2 \cdot (-4) = -48$. Thus $\det M = -48$.

Note that:

$$\det M = \det \begin{bmatrix} 1 & 3 \\ -1 & 3 \end{bmatrix} \cdot \det \begin{bmatrix} 2 & -1 \\ -4 & -2 \end{bmatrix} = 6 \cdot (-8) = -48$$

Indeed, it is true in general (and can be seen by row operations) that if a matrix is written in block form:

$$\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \qquad \text{or} \qquad \left[\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right]$$

(with A and C being square blocks) then its determinant is $\det(A)\det(C)$.